A Minimum Sobolev Norm Numerical Technique for PDEs

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We present a method for the numerical solution of PDEs based on finding solutions that minimize a certain Sobolev norm. Fairly standard compactness arguments establish convergence. The method prefers that the PDE is presented in first order form. A single short Octave code is used to solve problems that range from first-order Maxwell’s equations to fourth-order biharmonic problems on complicated geometries. The method is high-order convergent even on complex curved geometries.

Our method has its roots in generalized Birkhoff interpolation. Let $x_i$ denote $N$ points in $\mathbb{R}^d$. Let $f$ be an unknown function from $\mathbb{R}^d$ to $\mathbb{R}^q$. Given $N$ point-wise (possibly vector-valued) linear observations of $f$:

$$g(x_i) = \sum_{j \in \mathbb{N}^d} A_j(x_i) \partial_j^{\|j\|_1} f(x_i), \quad (1)$$

the problem is to compute an approximation to $f$.

The above problem is well-posed for some special choices of the matrix coefficients $A_j$ and points $x_i$ in the sense that, as $N$ approaches infinity, only the true solution can satisfy all the observations. In particular the numerical solution of linear PDEs can be posed as generalized Birkhoff interpolation problems. The classical approach to the above problem is to expand the unknown solution as a finite linear combination of basis functions, such that the constraints (1) become a system of square (or skinny) equations for the unknown coefficients. Then the (least-squares) solution of these equations is taken to be the computed solution for the unknown function. Our approach is almost the same, except that we pick more expansion coefficients so that we obtain a *fat* system of linear equations. As our solution we pick the one that minimizes a certain Sobolev norm. Assuming the true solution satisfies all the interpolation constraints and has a finite Sobolev norm, we can establish (see [1]) for the special case of classical interpolation) that
there is a uniform bound on the Sobolev norm of our computed solution independent of the number of constraints. It then follows from standard compactness arguments that our computed solution will converge to the true solution as the number of interpolation conditions increase.